Further mathematics
Higher level
Paper 2

Friday 20 May 2016 (morning)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

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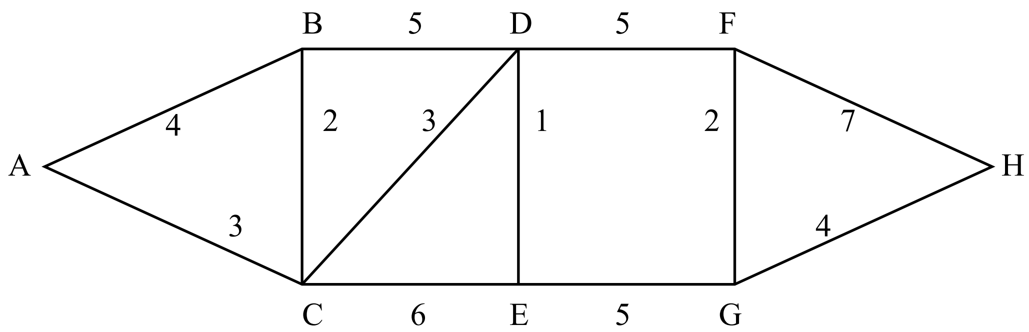
Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

Consider the following weighted graph.



(a) Determine whether or not the graph is Eulerian. [2]

(b) Determine whether or not the graph is Hamiltonian. [2]

(c) Use Kruskal's algorithm to find a minimum weight spanning tree and state its weight. [6]

(d) Deduce an upper bound for the total weight of a closed walk of minimum weight which visits every vertex. [2]

(e) Explain how the result in part (b) can be used to find a different upper bound and state its value. [2]

2. [Maximum mark: 17]

(a) Use l'Hôpital's rule to show that $\lim_{x \to \infty} \frac{x^3}{e^x} = 0$. [3]

The random variable X has probability density function given by

$$f(x) = \begin{cases} xe^{-x}, \text{ for } x \geq 0, \\ 0, \text{ otherwise} \end{cases}.$$

(b) (i) Find $E(X^2)$.

(ii) Show that $\text{Var}(X) = 2$. [10]

(c) State the central limit theorem. [2]

A sample of size 50 is taken from the distribution of X.

(d) Find the probability that the sample mean is less than 2.3. [2]

3. [Maximum mark: 15]

A circle C passes through the point $(1, 2)$ and has the line $3x - y = 5$ as the tangent at the point $(3, 4)$.

(a) Find the coordinates of the centre of C and its radius. [9]

(b) Write down the equation of C. [1]

(c) Find the coordinates of the second point on C on the chord through $(1, 2)$ parallel to the tangent at $(3, 4)$. [5]

4. [Maximum mark: 23]

Consider the differential equation $\frac{dy}{dx} = \frac{x}{y}$, where $y \neq 0$.

(a) Find the general solution of the differential equation, expressing your answer in the form $f(x, y) = c$, where c is a constant. [3]

(b) (i) Hence find the particular solution passing through the points $\left(1, \pm\sqrt{2}\right)$.

(ii) Sketch the graph of your solution and name the type of curve represented. [5]

(c) (i) Write down the particular solution passing through the points $(1, \pm 1)$.

(ii) Give a geometrical interpretation of this solution in relation to part (b). [3]

(d) (i) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$, where $xy \neq 0$.

(ii) Find the particular solution passing through the point $\left(1, \sqrt{2}\right)$.

(iii) Sketch the particular solution.

(iv) The graph of the solution only contains points with $|x| > a$.
Find the exact value of a, $a > 0$. [12]

5. [Maximum mark: 19]

(a) The sequence $\{u_n : n \in \mathbb{Z}^+\}$ satisfies the recurrence relation $2u_{n+2} - 3u_{n+1} + u_n = 0$, where $u_1 = 1$, $u_2 = 2$.

(i) Find an expression for u_n in terms of n.

(ii) Show that the sequence converges, stating the limiting value. [9]

(b) The sequence $\{v_n : n \in \mathbb{Z}^+\}$ satisfies the recurrence relation $2v_{n+2} - 3v_{n+1} + v_n = 1$, where $v_1 = 1$, $v_2 = 2$.
Without solving the recurrence relation prove that the sequence diverges. [3]

(c) The sequence $\{w_n : n \in \mathbb{N}\}$ satisfies the recurrence relation $w_{n+2} - 2w_{n+1} + 4w_n = 0$, where $w_0 = 0$, $w_1 = 2$.

(i) Find an expression for w_n in terms of n.

(ii) Show that $w_{3n} = 0$ for all $n \in \mathbb{N}$. [7]

6. [Maximum mark: 19]

Consider the set $J = \left\{a + b\sqrt{2} : a, b \in \mathbb{Z}\right\}$ under the binary operation multiplication.

(a) Show that J is closed. [2]

(b) State the identity in J. [1]

(c) Show that

(i) $1 - \sqrt{2}$ has an inverse in J;

(ii) $2 + 4\sqrt{2}$ has no inverse in J. [5]

(d) Show that the subset, G, of elements of J which have inverses, forms a group of infinite order. [7]

(e) Consider $a + b\sqrt{2} \in G$, where $\gcd(a, b) = 1$,

(i) Find the inverse of $a + b\sqrt{2}$.

(ii) Hence show that $a^2 - 2b^2$ divides exactly into a and b.

(iii) Deduce that $a^2 - 2b^2 = \pm 1$. [4]

7. [Maximum mark: 19]

Consider the functions $f_n(x) = \sec^n(x), |x| < \frac{\pi}{2}$ and $g_n(x) = f_n(x)\tan x$.

(a) Show that

(i) $\frac{\mathrm{d}f_n(x)}{\mathrm{d}x} = ng_n(x)$;

(ii) $\frac{\mathrm{d}g_n(x)}{\mathrm{d}x} = (n+1)f_{n+2}(x) - nf_n(x)$. [5]

(b) (i) Use these results to show that the Maclaurin series for the function $f_5(x)$ up to and including the term in x^4 is $1 + \frac{5}{2}x^2 + \frac{85}{24}x^4$.

(ii) By considering the general form of its higher derivatives explain briefly why all coefficients in the Maclaurin series for the function $f_5(x)$ are either positive or zero.

(iii) Hence show that $\sec^5(0.1) > 1.02535$. [14]

8. [Maximum mark: 24]

The set of all permutations of the list of the integers $1, 2, 3 \ldots n$ is a group, S_n, under the operation of composition of permutations.

(a) (i) Show that the order of S_n is $n!$;

(ii) List the 6 elements of S_3 in cycle form;

(iii) Show that S_3 is not Abelian;

(iv) Deduce that S_n is not Abelian for $n \geq 3$. [9]

(b) Each element of S_4 can be represented by a 4×4 matrix. For example, the cycle $(1\ 2\ 3\ 4)$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ acting on the column vector } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

(i) Write down the matrices $\boldsymbol{M}_1, \boldsymbol{M}_2$ representing the permutations $(1\ 2), (2\ 3)$, respectively;

(ii) Find $\boldsymbol{M}_1 \boldsymbol{M}_2$ and state the permutation represented by this matrix;

(iii) Find $\det(\boldsymbol{M}_1), \det(\boldsymbol{M}_2)$ and deduce the value of $\det(\boldsymbol{M}_1 \boldsymbol{M}_2)$. [7]

(c) (i) Use mathematical induction to prove that
$(1\ n)(1\ n-1)(1\ n-2)\ldots(1\ 2) = (1\ 2\ 3 \ldots n)\ n \in \mathbb{Z}^+, n > 1$.

(ii) Deduce that every permutation can be written as a product of cycles of length 2. [8]